

The Output Tracking Control of a Nonlinear Non-Minimum Phase System

Monia CHARFEDDINE^{#1}, Khalil JOUILI^{*2}, Naceur BENSADJ BRAIEK ^{#3}

Laboratory of Advanced Systems

Polytechnic School of Tunisia (EPT), B.P. 743, 2078 Marsa, Tunisia

¹monia.charfeddine@gmail.com

²khalil.jouili@gmail.com

³naceur.benhadj@ept.rnu.tn

Abstract— In this paper, we study the tracking control problem for a nonlinear system in which the zero dynamics may be unstable. The objective here is to approximate the nonlinear system by another singularly perturbed system by using a small singular perturbation parameter ε . The proposed approach is based on the methodology of exact input output linearization control, the integrator backstepping approach and the theory of singularly perturbed systems. The performance of the suggested approach is evaluated in an illustrative inverted cart-pendulum example.

Keywords—Non minimum phase system, integrator backstepping approach, input output linearization, singular perturbed system, tracking control.

I. INTRODUCTION

In the tracking control theory, the input output linearization is one of the most available methods [1]. If the nonlinear system has a stable zero dynamics, the input output linearization technique linearizes the nonlinear system by using a state feedback and a coordinate transformation [1], [2]-[3]. Thus, the linear system theory can be employed to design a controller achieving the desired control performance for the linearized closed loop transfer behavior. On the other hand, if the zero dynamics is unstable, the nonlinear system is called a non minimum phase. Thus, the standard input output linearization leads to an unstable closed loop system.

Consequently, the input output linearization of a non minimum phase system has attracted a lot of attention to enlarge the class of nonlinear systems where an input output linearization can be used [4]-[5]. In this contribution, Kravaris and Soroush have developed several results on the approximate linearization of non minimum phase systems [6], [7], [8]-[9]. For instance, in [8]-[9], the system output is differentiated as many times as the order of the system where the input derivatives that appear in the control law are set to zero when computing the state feedback input. In [10], the system input output feedback is first linearized. Then, the zero dynamics is factorized into stable and unstable parts. The unstable part is approximately linear and independent of the coordinates of the stable part. Moreover, an original technique of control based on an approximation of the method of exact input-output linearization was proposed in the works of Hauser and al [11]. In [12]-[13], the approximation presented in [11] is used to improve the desired control performance.

In this paper, we address the problem of tracking control of a single input single output of non minimum phase nonlinear systems. The idea here is to transform the given system into Brynes-Isidori normal form, then to use the singular perturbed theory in which a time-scale separation is artificially introduced through the use of a state feedback with a high-gain for the linearized part. The integrator backstepping approach is introduced to generate a reference trajectory for stabilizing the internal dynamics. The stability analysis for the proposed approach is based on the results of the singular-perturbation theory [14].

The present paper is organized as follows: in Section II some mathematical preliminaries are presented. The control law design and stability analysis are given in section III. Section IV gives the inverted cart-pendulum to illustrate the effectiveness of the proposed approach. Finally, some concluding remarks are provided in Section V.

II. SCOPE AND MATHEMATICAL PRELIMINARIES

In this paper, we consider a single input single output nonlinear system (SISO) of the form:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \quad (1)$$

where $x \in \mathfrak{R}^n$ is the n-dimensional state variables, $u \in \mathfrak{R}$ is a scalar manipulate input and $y \in \mathfrak{R}$ is a scalar output. $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are smooth functions describing the system dynamics.

A. Input output linearization

The input output linearization is based on two concepts: the concept of relative degree and the concept of state transformation.

The relative degree r of the system (1) is defined as the number of derivation of the output y needed to appear in the input u , such as $\forall x \in \mathfrak{R}^n$:

$$\begin{cases} L_f^k h(x) = 0 & \forall 1 \leq k \leq r-1 \\ L_g L_f^{(r-1)} h(x) \neq 0 \end{cases} \quad (2)$$

If $r \leq n$, then system (1) can be a feedback linearized into Brynes-Isidori normal form [15] using the following steps:

- **Step1:** We applied the following control law

$$u(x) = \frac{v - L_f^r h(x)}{L_g L_f^{r-1} h(x)} \quad (3)$$

with $v = y^{(r)}$

This control law compensates the nonlinearities in the input-output behavior.

- **Step 2:** we use the state transformation $\xi(x) = \Phi(x)$, given by:

$$\xi(x) = [h(x) \quad L_f h(x) \quad \dots \quad L_f^{r-1} h(x) \quad \eta_{n-r} \quad \dots \quad \eta_{n-r}]^T \quad (4)$$

The system (1) can be written as:

$$\begin{cases} \dot{\xi}_i = \xi_{i+1} & i = 1, \dots, r-1 \\ \dot{\xi}_r = v = a(x) + b(x)u \\ \dot{\eta} = Q(\xi, \eta) \\ y = \xi_1 \end{cases} \quad (5)$$

where η is the state vector of the internal dynamics.

B. Integrator backstepping approach

In this section, we consider the nonlinear system (1) which is written in the strict-feedback form given by [16], [17]-[18]:

$$\begin{cases} \dot{x}_i = f_i(x_1, x_2, \dots, x_i) + g_i(x_1, x_2, \dots, x_i)x_{i+1}, \quad \forall i \in \{1, 2, \dots, n-1\} \\ \dot{x}_n = f_n(x_1, x_2, \dots, x_n) + g_n(x_1, x_2, \dots, x_n)u \\ y = h(x_1) \end{cases} \quad (6)$$

where $x_i(t)$, $u(t)$ and $y(t)$ are the i th state variables, the system input and output are all assumed to be available for measurement; $f_i(\cdot)$ and $g_i(\cdot)$, $i=0, \dots, n$, are smooth nonlinear functions and $g_i \neq 0$.

The aim of the control is the trajectory tracking of the output y of the system (6), an error base e_i , $\forall i \in \{1, \dots, n\}$ is created as the difference between all the system states and their reference states $x_{i\text{ref}}$.

$$e_i = x_i - x_{i\text{ref}} \quad \forall i \in \{1, 2, \dots, n\} \quad (7)$$

The most popular method for nonlinear systems in the strict feedback form of (6) is the integrator backstepping approach developed in [16]-[19]. This is a Lyapunov-based approach in which the global convergence of the output tracking error is guaranteed in the absence of the model error. If this approach were applied to (6), then it would be possible to create a generator of trajectory for the state vector x in the form:

$$\begin{cases} x_{1\text{ref}} = h^{-1}(y_{\text{ref}}) \\ x_{2\text{ref}} = \frac{1}{g_1}(-f_1 + \dot{x}_{1\text{ref}} - \lambda_1 e_1) \\ x_{(i+1)\text{ref}} = \frac{1}{g_i}(-f_i + \dot{x}_{i\text{ref}} - g_{i-1}e_{i-1} - \lambda_i e_i), \quad \forall i \in \{1, 2, \dots, n-1\} \end{cases} \quad (8)$$

and the control law is as follow:

$$u = \frac{1}{g_n}(-f_n + \dot{x}_{n\text{ref}} - g_{(n-1)}e_{(n-1)} - \lambda_n e_n) \quad (9)$$

where $y_{\text{ref}}(t)$ is a reference trajectory at least C^n , h is a bijective function and h^{-1} is C^n .

Using (8) and (9), then the error dynamics equations are as follows:

$$\begin{cases} \dot{e}_1 = g_1 e_2 - \lambda_1 e_1 \\ \dot{e}_i = g_i e_{i+1} - g_{i-1} e_{i-1} - \lambda_i e_i, \quad \forall i \in \{2, \dots, n-1\} \\ \dot{e}_n = g_{n-1} e_{n-1} - \lambda_n e_n \end{cases} \quad (10)$$

It is easy to determine that the equilibrium point $e=0$ is the unique solution of (10).

In order to illustrate the stability of the origin $e=0$ of system (10), the following exponential stability theorem is introduced.

Theorem 1 [14]: Given system (1), if there exists a Lyapunov function $V(x)$ and positive constants χ_1, χ_2 and χ_3 such that $\chi_1 \|x\|^2 \leq V(x) \leq \chi_2 \|x\|^2$ and $\dot{V}(x) \leq -\chi_3 \|x\|^2$, then the origin is exponentially stable.

Consider the following Lyapunov function

$$V = \frac{1}{2} \sum_{i=1}^n e_i^2 \quad (11)$$

Therefore, the Lyapunov derivative \dot{V} is

$$\dot{V} = - \sum_{i=1}^n \lambda_i e_i^2 \quad \text{with } \lambda_i > 0, \forall i \in \{1, 2, \dots, n\} \quad (12)$$

So, the origin $e=0$ of system (10) is globally exponential stable

C. Singular perturbed system

A singularly perturbed system is one that exhibits a two-timescale behavior, i.e. it has a slow and fast dynamics and is modeled as follows [20]-[21]:

$$\dot{\eta} = F_1(\eta, \xi, u, \varepsilon) \quad \eta(0) = \eta_0 \quad (13a)$$

$$\varepsilon \dot{\xi} = F_2(\eta, \xi, u, \varepsilon), \quad \xi(0) = \xi_0 \quad (13b)$$

$$y = h(x) \quad (13c)$$

where $\eta \in \mathfrak{R}^m$ and $\xi \in \mathfrak{R}^p$ are respectively the slow and fast variables and $\varepsilon > 0$ is a small positive parameter. The functions F_1 and F_2 are assumed to be continuously differentiable. η_0 and ξ_0 are respectively the initial conditions of the vectors η and ξ . If $\varepsilon \rightarrow 0$, the dynamics of ξ acts quickly and leads to a time-scale separation. Such a separation can either represent the physics of the system or can be artificially created by the use of high-gain controllers.

As $\varepsilon \rightarrow 0$, ξ can be approximated by its quasi-steady state $\bar{\xi} = \psi(\eta, u)$ obtained by solving $f_1(\eta, \xi, 0) + g_1(\eta, \xi, 0)u = 0$.

So, the reduced (slow) system is given by:

$$\begin{aligned} \dot{\eta} &= f_2(\eta, \psi(\eta, u), 0) + g_2(\eta, \psi(\eta, u), 0)u \\ &= \bar{F}_2(\eta, u) \end{aligned} \quad (14)$$

Note that the reduced system (14) is not necessarily affine in input.

In the next theorem we establish the exponential stability of the singular perturbed system (13).

Theorem 2 [14]: Assume that the following conditions are satisfied:

- The origin is an equilibrium point for (13),

- $\psi(\eta, u)$ has a unique solution,
- The functions f_1, f_2, g_1, g_2, ψ and their partial derivatives up to order 2 are bounded for ξ in the neighborhood of $\bar{\xi}$,
- The origin of the boundary-layer system (13) is exponentially stable for all η ,
- The origin of the reduced system (14) is exponentially stable.

Then, there exists $\varepsilon^* > 0$ such that, for all $\varepsilon < \varepsilon^*$, the origin of (13) is exponentially stable.

III. CONTROL LAW DESIGN

In this section, an approach to the tracking control problem of the nonlinear non minimum phase system is proposed based on a singular perturbed theory and a combination of backstepping and input output linearization. In particular, it is shown that the closed-loop system can be described as an interconnection of two subsystems: the reduced subsystem and the boundary-layer subsystem. The stability analysis of the proposed approach is provided using the results of the singular perturbation theory [19].

A. Boundary Layer subsystem

Consider the nonlinear system described by (1), then we apply the control law (3) which is given by:

$$v = y^{(r)} = y_{ref}^{(r)} + \sum_{i=0}^{r-1} \bar{k}_{i+1} (y_{ref} - y)^{(i)} \quad \text{with} \quad \bar{k}_{i+1} = \frac{k_{i+1}}{\varepsilon^{r-i}} \quad (15)$$

where $y_{ref}(t)$ is the reference trajectory for the output, $\varepsilon \rightarrow 0$ a small positive parameter, and $k_i > 0, \forall i \in \{1, 2, \dots, n-1\}$ are the coefficients of a Hurwitz polynomial [14].

and the internal dynamics are given by:

$$\dot{\eta} = Q(\eta, y, \dot{y}, \dots, y^{(r-1)}) \quad (16)$$

Under the assumption that the gains \bar{k}_i are chosen large, such as for any choice of $\varepsilon > 0$, the closed loop is stable and ε can be used as a single tuning parameter, the system (15)-(16) can be written in the form of a singular perturbed system (13). So the fast state can be defined by:

$$\xi_i = \varepsilon^{i-1} y^{(i-1)}, \quad i = 1, \dots, r \quad (17)$$

If we replace (17) by (16), we obtain

$$\dot{\eta} = Q(\eta, \xi, \varepsilon), \quad \eta(0) = \eta_0 \quad (18)$$

and also by (15), such that

$$\varepsilon \dot{\xi}_r = \varepsilon^r y_{ref}^{(r)} + \sum_{i=0}^{r-1} k_{i+1} (\xi_{(i+1)ref} - \xi_{i+1}) \quad (19)$$

with $\xi_{ref} = [y_{ref} \quad \varepsilon \dot{y}_{ref} \quad \varepsilon^2 \ddot{y}_{ref} \quad \dots \quad \varepsilon^r y_{ref}^{(r-1)}]^T$

thus, (16) can be written as follows:

$$\begin{cases} \varepsilon \dot{\xi}_i = \xi_{i+1}, & i = 1, \dots, r-1 \\ \varepsilon \dot{\xi}_r = \varepsilon^r y_{ref}^{(r)} + \sum_{i=0}^{r-1} k_{i+1} (\xi_{(i+1)ref} - \xi_{i+1}) \end{cases} \quad (20)$$

B. Reduced subsystem

As the tuning parameter ε is small, so the quasi-steady-state assumption can be introduced. Thus, the reduced subsystem (QSS subsystem) when setting to zero in (18).

Letting $\varepsilon \rightarrow 0$ in (17)

$$\begin{cases} \xi_1 = y \\ \xi_2 = \dots = \xi_r = 0 \end{cases} \quad (21)$$

we use this result and let $\varepsilon \rightarrow 0$ in the last equation of (20), we obtain

$$\begin{aligned} \varepsilon \dot{\xi}_r &= \varepsilon^r y_{ref}^{(r)} + \sum_{i=0}^{r-1} k_{i+1} (\xi_{(i+1)ref} - \xi_{i+1}) = 0 \\ &\Rightarrow k_1 (\xi_{1ref} - \xi_1) = 0 \\ &\Rightarrow \xi_{1ref} = \xi_1 \end{aligned} \quad (22)$$

Therefore, when $\varepsilon \rightarrow 0$:

$$\xi = \bar{\xi} = [y_{ref} \quad 0 \quad \dots \quad 0]^T \quad (23)$$

The vector $\bar{\xi}$ is the quasi-steady-state value of ξ .

The internal dynamics depends on the output y , its derivatives $\dot{y}, \dots, y^{(r-1)}$ and the small parameter ε , such as:

$$\begin{aligned} \dot{\eta} &= Q(\eta, \xi, \varepsilon) \\ &= Q(\eta, \dot{y}, \dots, y^{(r-1)}, \varepsilon), \quad \eta(0) = \eta_0 \end{aligned} \quad (24)$$

Under the quasi-steady-state QSS assumption, the output y tends to y_{ref} and the derivatives $\dot{y}, \dots, y^{(r-1)}$ tend to their references $\dot{y}_{ref}, \dots, y_{ref}^{(r-1)}$. Then, the internal dynamics is written by

$$\dot{\eta} = Q(\eta, \dot{y}_{ref}, \dots, y_{ref}^{(r-1)}, \varepsilon), \quad \eta(0) = \eta_0 \quad (25)$$

Thus, the reference trajectory $(y_{ref}, \dot{y}_{ref}, \dots, y_{ref}^{(r-1)})$ will be used for the stabilization of the internal dynamics.

In general, the input output linearization techniques decouple between the input output behavior y and the internal dynamics η . On the other hand, the quasi-steady-state (QSS) assumption decouples between the internal dynamics η and the input output behavior y . Thus, y does not have any effect on η . Therefore, the reference trajectory y_{ref} is used for the control of the internal dynamics. Thus, the Boundary Layer subsystem (20) and the reduced subsystem (QSS subsystem) (25) can be manipulated separately.

Firstly, we define a novel state vector $\bar{\eta} = [\eta_1 \quad \dots \quad \eta_{n-r} \quad y_{ref} \quad \dots \quad y_{ref}^{(r-1)}]^T$ such as the reduced subsystem (QSS subsystem) (25) can be written by

$$\dot{\bar{\eta}} = \bar{Q}(\bar{\eta}, u_{QSS}) \quad \text{with} \quad u_{QSS} = y_{ref}^{(r)} \quad (26)$$

Note that it is important to include additional states $y_{ref}, \dot{y}_{ref}, \dots, y_{ref}^{(r-1)}$ since they are considered as independent variables, and the last derivative $u_{QSS} = y_{ref}^{(r)}$ is considered as the control law for (26).

To determine u_{QSS} , we need the assumption that the internal dynamics (26) is written in the following strict feedback form:

$$\dot{\bar{\eta}} = \begin{cases} \dot{\eta}_1 = \alpha_1(\eta_1) + \beta_1(\eta_1)\eta_2 \\ \dot{\eta}_2 = \alpha_2(\eta_1, \eta_2) + \beta_2(\eta_1, \eta_2)\eta_3 \\ \vdots \\ \dot{\eta}_{n-r} = \alpha_{n-r}(\eta_1, \dots, \eta_{n-r}) + \beta_{n-r}(\eta_1, \dots, \eta_{n-r})\eta_{n-r+1} \\ \dot{\eta}_{n-r+1} = \alpha_{n-r+1}(\eta_1, \dots, \eta_{n-r+1}) + \beta_{n-r+1}(\eta_1, \dots, \eta_{n-r+1})\eta_{n-r+2} \\ \vdots \\ \dot{\eta}_n = \alpha_n(\eta_1, \dots, \eta_n) + \beta_n(\eta_1, \dots, \eta_n)u_{QSS} \end{cases} \quad (27)$$

Then, we use the integrator backstepping approach to determine the reference trajectory y_{ref} and the control law u_{QSS} . So, we define $y_{QSS} = \varphi(\eta_1)$ as a virtual output for the subsystem (26) and y_{QSSref} are the reference trajectory for the output y_{QSS} . By referring to the equations (8) and (9), we obtain the following trajectory generator:

$$\eta_{ref} = \begin{cases} \eta_{1ref} = \varphi^{-1}(y_{sref}) \\ \eta_{2ref} = \frac{1}{\beta_1}(-\alpha_1 + \dot{\eta}_{1ref} - \lambda_1 \tilde{\eta}_1) \\ \eta_{(i+1)ref} = \frac{1}{\beta_i}(-\alpha_i + \dot{\eta}_{iref} - \beta_{i-1} \tilde{\eta}_{i-1} - \lambda_i \tilde{\eta}_i), \quad i=3, \dots, n-1 \end{cases} \quad (28)$$

where $\tilde{\eta} = \eta - \eta_{ref}$

and the control law is given by

$$u_{QSS} = \frac{1}{\beta_n}(-\alpha_n + \dot{\eta}_{n-1ref} - \beta_{n-1} \tilde{\eta}_{n-1} - \lambda_n \tilde{\eta}_n) \quad (29)$$

C. Stability analysis

In this section, we use the theorem 2 of exponential stability of singular perturbed system to analyze the stability of the closed loop system. If both the reduced and the boundary layer subsystems are exponentially stable, then the combination is also exponentially stable. The following steps will be used to prove the stability of the proposed approach:

1) Exponential stability of the Boundary Layer subsystem

Let us consider the error vector given by

$$\tilde{\xi} = \xi - \xi_{ref} \quad (30)$$

Then, the boundary layer subsystem (20) becomes:

$$\varepsilon \dot{\tilde{\xi}} = \varepsilon \tilde{\xi} - \varepsilon \dot{\xi}_{ref} \quad (31)$$

$$= \begin{bmatrix} \tilde{\xi}_1 & \tilde{\xi}_2 & \dots & \sum_{i=0}^{r-1} k_{i+1} \tilde{\xi}_{i+1} \end{bmatrix}^T$$

Letting $\tau = \frac{t}{\varepsilon}$ yield:

$$\frac{d\tilde{\xi}}{d\tau} = A\tilde{\xi} \quad (32)$$

with A is defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_{n-r} \end{bmatrix}$$

Using the theorem1, the origin $\tilde{\xi} = 0$ is exponentially stable, and the Lyapunov function is

$$V = \frac{1}{2} \tilde{\xi}^T \Psi \tilde{\xi}_1 \quad (33)$$

where $A^T \Psi + \Psi A = -Q$ and Q is a matrix defined positive

2) Exponential stability of the reduced subsystem

The stability of the reduced subsystem is provided by using the integrator backstepping approach. So the Lyapunov function is given by:

$$V = \frac{1}{2} \sum_{i=1}^{n-1} \tilde{\eta}_i^2 \quad (34)$$

3) Global exponential stability

Using the theorem 2, we can conclude that the origin of (1) is exponentially stable. Although all the conditions of the theorem 1 are satisfied such that

- The origin ($\eta = 0, \xi = 0$ and $y_{ref} = 0$) is an equilibrium point for the subsystems (20) and (26)
- $\bar{Q}(\bar{\eta}, u_{QSS})$ has a unique solution $\eta^* = [0 \dots 0 \ y_{ref} \ 0 \dots 0]$
- Furthermore, as a result of the integrator backstepping approach, y_{ref} is a function of η
- Q and its partial derivatives up to order 2 are bounded for η in the neighborhood of $\bar{\eta}$
- The origin of the boundary layer system (20) is exponentially stable $\forall \eta$
- The origin of the reduced system (26) is exponentially stable

IV. ILLUSTRATIVE EXAMPLE

Consider the cart-inverted pendulum illustrated in fig. 1. The cart must be moved using the force $u(t)$ so that the pendulum remains in the upright position as the cart tracks varying positions at the desired time. The differential equations describing the motion are [22]:

$$\begin{cases} (M+m)\ddot{y}_p + ml\ddot{\theta} \cos\theta + ml\dot{\theta}^2 \sin\theta = u \\ l\ddot{\theta} - \ddot{y}_p \cos\theta - g \sin\theta = 0 \end{cases} \quad (35)$$

where θ is the angle of the pendulum, y_p is the displacement of the cart, and u is the control force, parallel to the rail, applied to the cart.

The numerical parameters of the inverted pendulum system are $M=0.455\text{kg}$, $m=0.21\text{kg}$, $l=0.355$ and $g=9.81\text{m/s}^2$

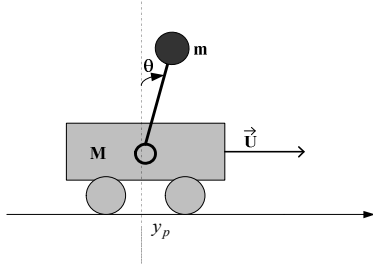


Fig. 1. inverted pendulum system

The system can be put into the state space form as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{l}(g \sin x_1 - \dot{x}_4 \cos x_1) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \frac{u + m(lx_2^2 + g \cos x_1) \sin x_1}{M + m(\sin x_1)^2} \\ y = x_1 \end{cases} \quad (36)$$

where $x = (x_1 \ x_2 \ x_3 \ x_4)^T = (\theta \ \dot{\theta} \ y \ \dot{y})^T$. The relative degree of the system is equal to $r=2$ which is strictly lower than the system dimension $n=4$.

Applying the procedure of input output linearization to the system (36) of the inverted cart-pendulum, the Boundary Layer system is given by:

$$\begin{cases} \dot{\xi}_1 = x_1 \\ \dot{\xi}_2 = x_2 \end{cases} \quad (37)$$

its control is given by:

$$u = lm(lx_2^2 + g \cos x_1) \sin x_1 - \frac{l(M + m(\sin x_1)^2)}{\cos x_1} v \quad (38)$$

with $v = y_{ref}^{(2)} + \frac{k_2}{\varepsilon}(\dot{y}_{ref} - \dot{y}) + k_1(y_{ref} - y)$

and the internal dynamics is given by

$$\eta(x) = \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 - \frac{\cos x_1}{l} x_2 \end{bmatrix} \quad (38)$$

Under the QSS assumption that $\dot{\theta} = \ddot{\theta} = v = 0$ and

$\theta \rightarrow \theta_{ref} \cong 0$, $\sin \theta = \theta$ and $\cos \theta = 1$.

Using the equation (26), the reduced subsystem can be written as:

$$\dot{\bar{\eta}}(x) = \begin{bmatrix} \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = g \eta_3 \\ \dot{\eta}_3 = \eta_4 \\ \dot{\eta}_4 = u_{QSS} \end{bmatrix} \quad (39)$$

with $\eta_3 = y_{ref}$, $\eta_4 = \dot{y}_{ref}$ and $u_{QSS} = y_{ref}^{(2)}$

Thus, the integrated backstepping approach is used to compute the stabilizing input u_{QSS} and the reference trajectory

y_{ref} .

By referring to the equations (29) and (30), we obtain:

$$\eta_{ref} = \begin{cases} \eta_{1,ref} = 1 - e^{-t} \\ \eta_{2,ref} = e^{-t} - \lambda_1 e_1 \\ \eta_{3,ref} = \frac{1}{g}(-e^{-t} - (\lambda_1 + \lambda_2)e_2 - (1 - \lambda_1^2)e_1) \\ \eta_{4,ref} = \frac{1}{g}e^{-t} - (\lambda_1 + \lambda_2 + \lambda_3)e_3 \\ \quad + \frac{1}{g}(\lambda_1 \lambda_2 + \lambda_2^2 + \lambda_1^2 - 1 - g^2)e_2 + \frac{1}{g}(2\lambda_1 + \lambda_2 - \lambda_1^3)e_1 \end{cases} \quad (40)$$

and the stabilizing input u_{QSS} is given by

$$\begin{aligned} u_{QSS} = & \frac{1}{g}e^{-t} + \frac{1}{g}(\lambda_1^4 + 1 + g^2 - \lambda_2^2 - 3\lambda_1^2 - 2\lambda_1 \lambda_2)e_1 \\ & + \frac{1}{g}(2\lambda_1 + 3\lambda_2 - \lambda_1 \lambda_2^2 + \lambda_2 \lambda_1^2 - \lambda_1^3 - \lambda_2^3 + g^2(\lambda_1 + 2\lambda_2 + \lambda_3))e_2 \\ & + (\lambda_1 \lambda_2 + \lambda_3 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2 - g^2)e_3 \\ & + (-\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)e_4 \end{aligned} \quad (42)$$

The controller (38) based on the integrator backstepping approach is now implemented to the inverted cart-pendulum model described by equation (36).

As a result, in fig. 2, we show the evolution of the tracking trajectory compared to the desired one $y_{ref}(t)$. In this figure, indeed, there is a perfect agreement between the two trajectories. The internal dynamics is stable; it can clearly be seen in fig. 3. Figure 4 represents the evolution of the stabilizing control law. The dynamics of this control signal is quite satisfactory.

In fact, there is no unacceptable physical overshoot. One can also see the reduced response time in which the control law stabilizes the controlled variable. The tracking error between the reference and the trajectory is reduced. This shows the very interesting results given by the developed approach.

VI. ACKNOWLEDGMENT

The authors wish to thank the editor and the reviewers for their constructive comments that have improved the quality of this paper.

REFERENCES

- [1] A. Isidori, *Nonlinear control systems*, London: Springer-Verlag, 1995.
- [2] L. R. Hunt, R. Su and G. Mayer, "Global transformations of nonlinear systems", *IEEE Transaction On Automatic Control*, vol. 28, pp. 24-31, 1983.
- [3] H. Nijmeijer, A.J. Van der shaft. *Nonlinear dynamical control systems*, Berlin :Springer -Verlag, 1990.
- [4] F. Allgöwer and J. F. Doyle, "Approximate I/O-linearization of nonlinear systems", in *Nonlinear model based process control*, R. Berber and C. Kravaris, Eds. Kluwer Academic Publishers, Dordrecht, pp. 235-274, 1998.
- [5] G. O. Guardabassi and S. M. Savaresi, "Approximate linearization via feedback -an overview", *Automatica*, vol. 37, pp. 1-15, Jan. 2001.
- [6] C. Kravaris, P. Daoutidis, and R. A. Wright. "Output feedback control of non minimum-phase nonlinear processes", *Chemical Engineering Science*, vol. 49, pp. 2107-2122, Jul. 1994.
- [7] J. M. Kanter, M. Soroush, and W. D. Seider. "Continuous time, nonlinear feedback control of stable processes". *Industrial & Engineering Chemistry Research*, vol. 40, pp. 2069-2078, Apr. 2001.
- [8] C. Kravaris and P. Daoutidis."Output feedback controller realizations for open-loop stable nonlinear processes". *Proc. ACC'92*, 1992, p. 2576.
- [9] M. Soroush and C. Kravaris. "A continuous-time formulation of nonlinear model predictive control", *International Journal of Control*, vol. 63, pp. 121-146, 1996.
- [10] F. Allgöwer. "Approximate input-output linearization of nonminimum phase nonlinear systems", *Proc. ECC'97*, 1997, p. 604.
- [11] J. Hauser, S. Sastry and P. Kokotovic. "Nonlinear control via approximates input-output linearization: the ball and beam example", *IEEE Transaction on Automatic Control*, Vol. 37, pp. 392-398, Mar. 1992.
- [12] M. Charfeddine, K. Jouili, H. Jerbi and N. Benhadj Braiek, "Linearizing Control with a Robust Relative Degree Based on Lyapunov Function: Case of the Ball and Beam system", *International Review on Modelling and Simulations*, vol. 3, pp. 219-226, Apr. 2010
- [13] M. Charfeddine, K. Jouili, H. Jerbi and N. Benhadj Braiek, "Output Tracking Control Design for Non-Minimum Phase Systems: Application to the Ball and Beam Model", *International Review of Automatic Control*, Vol.4, pp. 47-55, Jan. 2011
- [14] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, New Jersey, 1996.
- [15] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems", *IEEE Transaction on Automatic Control*, Vol. 35, pp. 131-140, Feb. 1990.
- [16] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and adaptive control design*, Wiley: New York, 1995.
- [17] V. Andrieu, L. Praly, "Global asymptotic stabilization for non minimum phase non linear systems admitting a strict normal form". *IEEE Transactions on Automatic Control*, Vol. 53, pp. 1120-1132, Jun, 2008.
- [18] L. Deng and S. Gao, "The design for the controller of the linear inverted pendulum based on backstepping", *Proc. EMEIT'11*, 2011, p. 2892-2895.
- [19] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Transactions on Automatic Control*, vol. 36, pp. 1241-1253, Nov. 1991.
- [20] P.V. Kokotovic, H.K. Khalil and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*, Philadelphia: SIAM, 1999.
- [21] Luigi Glielmo and Martin Corless, "On output feedback control of singularly perturbed systems", *Applied Mathematics and Computation*, vol. 217, pp. 1053-1070, Oct. 2010.
- [22] S. Al-hiddabi, "Implementation of stable inversion control on a nonlinear non-minimum phase system: cart pendulum experiment", *Proc. SICE'05*, 2005.

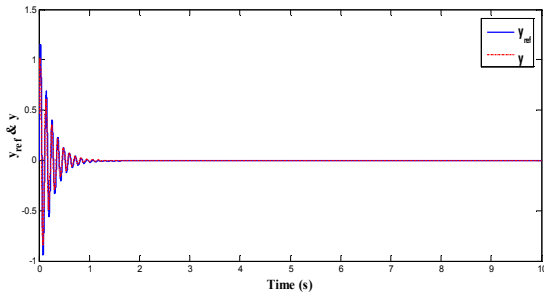


Fig. 2. Inverted pendulum system

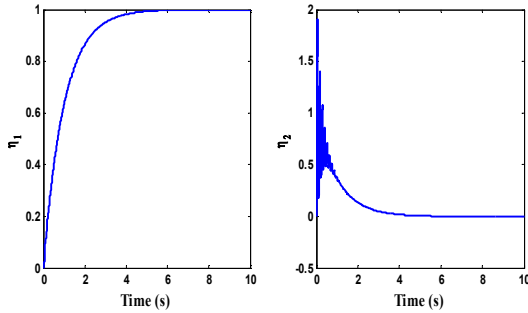


Fig. 3. Evolution of the internal dynamics states

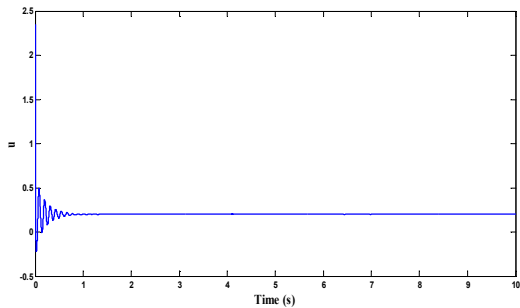


Fig. 4. Evolution of the nonlinear controller

V. CONCLUSIONS

In this paper, we propose a new nonlinear tracking control approach for non minimum phase systems which is based on the approximation of the non minimum system by another singular perturbed system

The method uses the input output linearization technique to cancel the nonlinearities of the external dynamics and to stabilize the internal dynamics by the integrator backstepping approach. A stability analysis of the proposed approach has been provided based on the singular perturbation theory.

The proposed approach has been used to design a globally exponential stabilizing controller for the inverted cart-pendulum system where the rotation angle has a low dynamics around zero.

The future research will also address the issue of estimating the internal dynamics for each sampling period and will develop the appropriate techniques to stabilize it.